Int. J. Solids Structures Vol. 28, No. 5, pp. 631–645, 1991
Printed in Great Britain.

0020-7683/91 $3.00+.00

COMPUTATION OF THE FUNDAMENTAL SOLUTION FOR SHALLOW SHELLS INVOLVING SHEAR DEFORMATION

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(*Received* 2 *April* 1990; *in revised form* 20 *October* 1990)

Abstract—The fundamental solution for the theory of doubly curved shallow shells involving shear deformation is obtained by means of Hörmander's operator and plane-wave decomposition methods. This solution has many important applications in the theoretical and numerical analyses for the shells. Methods of evaluating the fundamental solution obtained are also discussed in this paper, and numerical results are presented for the case of a concentrated normal force acting on infinite shells having positive, zero or negative Gaussian curvature.

INTRODUCTION

As is well known, fundamental solutions (or singular solutions) have many applications in the studies of mechanical problems. For example, they can be used to analyse the stress and displacement distributions in the neighborhood of the singular point which a concentrated force is applied to, and they can also be chosen as the kernels of the boundary integral equations in the Boundary Element Method which is widely used in engineering. Therefore, it is essential to find the fundamental solutions for different kinds of mechanical problems, especially for shell structures. For shallow thin shells with an arbitrary quadratic middle surface, the fundamental solutions have been obtained and discussed in detail (Flügge and Elling, 1972; Jahanshahi, 1964; Matsui and Matsuoka, 1978; Sanders, 1970; Simmonds and Bradley, 1976). However, investigations of shallow shells involving shear deformation are comparatively few because of the complexity of the governing equations. In the present paper, this kind of shell is investigated and the corresponding fundamental solutions are derived. These results are quite important for studies of shallow shells considering shear deformation.

In order to simplify the governing equations of this kind of shallow shell, Hörmander's operator method is used and a tenth order partial differential equation is obtained. Then, by using of plane-wave decomposition method, the problem of finding the fundamental solution of the partial differential equation can be further reduced to solving an ordinary differential equation. Therefore, the fundamental solution of an arbitrary doubly curved shallow shell involving shear deformation can be obtained in this way.

In this paper, the computational methods for the fundamental solutions obtained are also discussed in detail and numerical results are presented for the case of a concentrated normal force acting on infinite shells having positive, zero or negative Gaussian curvature.

DECOMPOSITION OF BASIC EQUATIONS

Consider a shallow shell with a quadratic middle surface given by

$$z = -\tfrac{1}{2}(k_1x^2+k_2y^2) \tag{1}$$

where k_1 and k_2 are principal curvatures of the shell in the x and y directions respectively. Therefore, according to the simplifications of shallow shell theory, the basic equations of the shallow shells involving shear deformation can be expressed as follows (Sih, 1977):

Equilibrium equations

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=p_x,\quad \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=p_y,$$

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}-(k_1N_x+k_2N_y)+p_z=0,\quad \frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=m_x,$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-Q_y=m_y \tag{2}$$

Stress–displacement relations

$$N_x=B\left[\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}+(k_1+\nu k_2)w\right],\quad N_y=B\left[\frac{\partial v}{\partial y}+\nu\frac{\partial u}{\partial x}+(k_2+\nu k_1)w\right],$$

$$N_{xy}=\tfrac{1}{2}(1-\nu)B\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\quad Q_x=C\left(\frac{\partial w}{\partial x}+\psi_x\right),\quad Q_y=C\left(\frac{\partial w}{\partial y}+\psi_y\right),$$

$$M_x=D\left(\frac{\partial \psi_x}{\partial x}+\nu\frac{\partial \psi_y}{\partial y}\right),\quad M_y=D\left(\frac{\partial \psi_y}{\partial y}+\nu\frac{\partial \psi_x}{\partial x}\right),\quad M_{xy}=\tfrac{1}{2}(1-\nu)D\left(\frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x}\right) \tag{3}$$

where u, v, w, ψ_x and ψ_y are five independent displacements, p_x, p_y, p_z, m_x and m_y are generalized distributive loads applied in different directions of the shell, B, C and D are tension, shear and bending stiffness respectively, and ν is Poisson's ratio.

Substituting (3) into (2), we can obtain the equilibrium equations expressed in displacements

$$[\mathbb{L}]\{\mathbb{U}\}=\{\mathbb{P}\} \tag{4}$$

where

$$\{\mathbb{U}\}=[u,v,w,\psi_x,\psi_y]^T,\quad \{\mathbb{P}\}=[p_x,p_y,p_z,m_x,m_y]^T \tag{5}$$

and $[\mathbb{L}]$ is a symmetrical differential operator matrix of order 5 × 5. Its elements are

$$L_{11}=\frac{B}{2}[(1+\nu)D_1^2+(1-\nu)\nabla^2],\quad L_{12}=\tfrac{1}{2}(1+\nu)BD_1D_2,$$

$$L_{13}=B(k_1+\nu k_2)D_1,\quad L_{14}=L_{15}=0,$$

$$L_{22}=\frac{B}{2}[(1+\nu)D_2^2+(1-\nu)\nabla^2],\quad L_{23}=B(k_2+\nu k_1)D_2,\quad L_{24}=L_{25}=0,$$

$$L_{33}=-C\nabla^2+B(k_1^2+k_2^2+2\nu k_1k_2),\qquad L_{34}=-CD_1,\quad L_{35}=-CD_2,$$

$$L_{44}=\frac{D}{2}[(1+\nu)D_1^2+(1-\nu)\nabla^2]-C,\quad L_{45}=\tfrac{1}{2}(1+\nu)DD_1D_2,$$

$$L_{55}=\frac{D}{2}[(1+\nu)D_2^2+(1-\nu)\nabla^2]-C \tag{6}$$

where D_1, D_2 and ∇^2 are partial differential operators given by

$$D_1=\frac{\partial}{\partial x},\quad D_2=\frac{\partial}{\partial y},\quad \nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=D_1^2+D_2^2. \tag{7}$$

Equations (4) can be decomposed by Hörmander's operator method (Hörmander, 1963). Define the following displacement functions

$$\{\Phi^*\} = [\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5]^T \tag{8}$$

which satisfy the relation

$$\{\mathbb{U}\} = [\mathbb{F}]\{\Phi^*\} \tag{9}$$

where differential operator matrix $[\mathbb{F}]$ can be obtained by the following method. Let

$$[\mathbb{L}][\mathbb{F}] = [\mathbb{I}]\mathscr{L} \tag{10}$$

where $[\mathbb{I}]$ is a unit matrix of order 5×5, and $\mathscr{L}$ is a differential operator given by the determinant of the operator matrix $[\mathbb{L}]$. From (10), we have

$$[\mathbb{F}] = [\mathbb{L}]^{-1}\mathscr{L}. \tag{11}$$

Therefore, $[\mathbb{F}]$ and $\mathscr{L}$ can be determined from the known operator matrix $[\mathbb{L}]$, and are listed at the end of this section. We notice that coupled equations (4) have now been reduced to a set of uncoupled equations for five displacement functions

$$\mathscr{L}(\Phi_j) = p_j/D \quad (j = 1, 2, \ldots, 5). \tag{12}$$

Let $\Phi(x, y)$ be the fundamental solution of the differential operator $\mathscr{L}$, i.e.

$$\mathscr{L}(\Phi) = \delta(x, y) \tag{13}$$

where $\delta(x, y)$ is the Dirac δ-function. Then the particular solutions to the set of differential equations (12) for arbitrary loading can be expressed in the form

$$\Phi_j(x, y) = \iint \Phi(x-\zeta, y-\eta)\frac{p_j}{D}\,\mathrm{d}\zeta\,\mathrm{d}\eta \quad (j = 1, 2, \ldots, 5). \tag{14}$$

Once $\Phi_j(x, y)$ is known, the displacements can be determined from (9) in terms of the displacement functions, and the generalized stress resultants can be obtained from relation (3):

$$\{\mathbb{T}\} = [\mathbb{G}]\{\mathbb{U}\} = [\mathbb{G}][\mathbb{F}]\{\Phi^*\} = [\mathbb{R}]\{\Phi^*\} \tag{15}$$

where

$$\{\mathbb{T}\} = [N_x, N_y, N_{xy}, Q_x, Q_y, M_x, M_y, M_{xy}]^T \tag{16}$$

and the elements of the differential operator matrix $[\mathbb{R}]$ are listed in the Appendix. The representations of the differential operator $\mathscr{L}$ and the symmetric operator matrix $[\mathbb{F}]$ can be derived according to the methods discussed before, and are listed below:

$$\mathscr{L} = \left[\nabla^8 - (1-v^2)\frac{B}{C}\nabla_k^4\tilde{L}_2\right]\tilde{L}_1 \tag{17}$$

$$F_{11} = \left[\frac{D}{B}\nabla^4\tilde{L}_4 - \frac{D}{C}\tilde{L}_2\tilde{L}_6\right]\tilde{L}_1, \quad F_{12} = \left[-\frac{D}{B}\frac{1+v}{1-v}\nabla^4 + \frac{D}{C}(k_1-k_2)^2\tilde{L}_2\right]\tilde{L}_1D_1D_2,$$

$$F_{13} = \frac{D}{C}\tilde{L}_1\tilde{L}_2\tilde{L}_8, \quad F_{14} = \tilde{L}_1\tilde{L}_8 D_1, \quad F_{15} = \tilde{L}_1\tilde{L}_8 D_2,$$

$$F_{22} = \left[\frac{D}{B}\nabla^4\tilde{L}_5 - \frac{D}{C}\tilde{L}_2\tilde{L}_7\right]\tilde{L}_1, \quad F_{23} = \frac{D}{C}\tilde{L}_1\tilde{L}_2\tilde{L}_9, \quad F_{24} = \tilde{L}_1\tilde{L}_9 D_1, \quad F_{25} = \tilde{L}_1\tilde{L}_9 D_2,$$

$$F_{33} = -\frac{D}{C}\nabla^4\tilde{L}_1\tilde{L}_2, \quad F_{34} = -\nabla^4\tilde{L}_1 D_1, \quad F_{35} = -\nabla^4\tilde{L}_1 D_2,$$

$$F_{44} = \tilde{L}_3\left(\tilde{L}_4 - \frac{2}{1-\nu}\frac{C}{D}\right) + \frac{2}{1-\nu}\frac{C}{D}\nabla^4 D_2^2, \quad F_{45} = -\frac{1+\nu}{1-\nu}\tilde{L}_3 D_1 D_2 - \frac{2}{1-\nu}\frac{C}{D}\nabla^4 D_1 D_2,$$

$$F_{55} = \tilde{L}_3\left(\tilde{L}_5 - \frac{2}{1-\nu}\frac{C}{D}\right) + \frac{2}{1-\nu}\frac{C}{D}\nabla^4 D_1^2 \tag{18}$$

where

$$\tilde{L}_1 = \nabla^2 - \frac{2}{1-\nu}\frac{C}{D}, \quad \tilde{L}_2 = \nabla^2 - \frac{C}{D}, \quad \tilde{L}_3 = \nabla^6 - (1-\nu^2)\frac{B}{C}\nabla_k^4,$$

$$\tilde{L}_4 = D_1^2 + \frac{2}{1-\nu}D_2^2, \quad \tilde{L}_5 = D_2^2 + \frac{2}{1-\nu}D_1^2, \quad \tilde{L}_6 = (k_1^2+k_2^2+2\nu k_1 k_2)D_1^2 + 2(1+\nu)k_1^2 D_2^2,$$

$$\tilde{L}_7 = (k_1^2+k_2^2+2\nu k_1 k_2)D_2^2 + 2(1+\nu)k_2^2 D_1^2, \quad \tilde{L}_8 = (k_1+\nu k_2)D_1^3 + [(2+\nu)k_1 - k_2]D_1 D_2^2,$$

$$\tilde{L}_9 = (k_2+\nu k_1)D_2^3 + [(2+\nu)k_2 - k_1]D_1^2 D_2, \quad \nabla_k^2 = k_2\frac{\partial^2}{\partial x^2} + k_1\frac{\partial^2}{\partial y^2} = k_2 D_1^2 + k_1 D_2^2. \tag{19}$$

FUNDAMENTAL SOLUTION

Now the problem of finding the fundamental solution of an arbitrary quadratic mid-surface shallow shell involving shear deformation can be reduced to solving eqn (13)

$$\mathscr{L}(\Phi) = \left[\nabla^8 - (1-\nu^2)\frac{B}{C}\nabla_k^4\tilde{L}_2\right]\tilde{L}_1\{\Phi(x,y)\} = \delta(x,y). \tag{20}$$

This is a tenth order partial differential equation, and can be solved by the plane-wave decomposition method (Gel'fand and Shilov, 1966).

Firstly, by expanding the right-hand side of eqn (20) into plane-waves in two-dimensional space, we obtain the equation

$$\mathscr{L}(\Phi) = -\frac{1}{4\pi^2}\int_0^{2\pi} |\omega_1 x + \omega_2 y|^{-2}\,\mathrm{d}\theta \tag{21}$$

where (ω_1, ω_2) are the coordinates of a point on the unit circle, i.e.

$$\omega_1 = \cos\theta, \quad \omega_2 = \sin\theta. \tag{22}$$

Let

$$\rho = \omega_1 x + \omega_2 y. \tag{23}$$

Consider the equation

$$\mathscr{L}(\phi) = -\frac{1}{4\pi^2}|\omega_1 x + \omega_2 y|^{-2} = -\frac{1}{4\pi^2}|\rho|^{-2}. \tag{24}$$

If a solution $\phi(\rho)$ which depends only on the ρ in eqn (24) could be obtained, the solution of eqn (21) can be written in the form

$$\Phi(x,y) = \int_0^{2\pi} \phi(\omega_1 x+\omega_2 y)\,\mathrm{d}\theta = \int_0^{2\pi} \phi(\rho)\,\mathrm{d}\theta. \tag{25}$$

Equation (25) is called the plane-wave representation of the fundamental solution. Therefore, the partial differential operators can be written as

$$\frac{\partial}{\partial x} = \omega_1 \frac{\mathrm{d}}{\mathrm{d}\rho}, \qquad \frac{\partial}{\partial y} = \omega_2 \frac{\mathrm{d}}{\mathrm{d}\rho}. \tag{26}$$

Applying these relations to the left-hand side of eqn (24), we can obtain the ordinary differential equation as follows:

$$\left[\frac{\mathrm{d}^4}{\mathrm{d}\rho^4}\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}-a_1^2\right)\left(\frac{\mathrm{d}^4}{\mathrm{d}\rho^4}-a_2\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}+a_3\right)\right]\phi(\rho) = -\frac{1}{4\pi^2}|\rho|^{-2} \tag{27}$$

where

$$a_1 = \left(\frac{2}{1-\nu}\frac{C}{D}\right)^{1/2}, \quad a_2 = (k_1\omega_2^2+k_2\omega_1^2)^2(1-\nu^2)\frac{B}{C}, \quad a_3 = a_2\frac{C}{D}. \tag{28}$$

Now the problem of solving eqn (20) is reduced to solving eqn (27). After four integrations of eqn (27), we have

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}-a_1^2\right)\left(\frac{\mathrm{d}^4}{\mathrm{d}\rho^4}-a_2\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}+a_3\right)\phi(\rho) = \frac{1}{8\pi^2}\rho^2 \ln|\rho|. \tag{29}$$

The particular solution of eqn (29) can be obtained by the method of variation of parameters (Ye and Xu, 1978)

$$\phi(\rho) = u_1\,\mathrm{e}^{r_1\rho}+u_2\,\mathrm{e}^{-r_1\rho}+u_3\,\mathrm{e}^{r_2\rho}+u_4\,\mathrm{e}^{-r_2\rho}+u_5\,\mathrm{e}^{r_3\rho}+u_6\,\mathrm{e}^{-r_3\rho} \tag{30}$$

where

$$\begin{aligned} r_1 &= a_1 \\ r_2 &= (\sqrt{a_3/4}+a_2/4)^{1/2}+i(\sqrt{a_3/4}-a_2/4)^{1/2} \\ r_3 &= (\sqrt{a_3/4}+a_2/4)^{1/2}-i(\sqrt{a_3/4}-a_2/4)^{1/2} \end{aligned} \tag{31}$$

$$\begin{aligned} u_1 &= -\frac{1}{16\pi^2 r_1(r_1^2-r_2^2)(r_1^2-r_3^2)}\int_\rho^{\infty} \sigma^2 \ln|\sigma|\,\mathrm{e}^{-r_1\sigma}\,\mathrm{d}\sigma \\ u_2 &= -\frac{1}{16\pi^2 r_1(r_1^2-r_2^2)(r_1^2-r_3^2)}\int_{-\infty}^{\rho} \sigma^2 \ln|\sigma|\,\mathrm{e}^{r_1\sigma}\,\mathrm{d}\sigma \\ u_3 &= -\frac{1}{16\pi^2 r_2(r_2^2-r_1^2)(r_2^2-r_3^2)}\int_\rho^{\infty} \sigma^2 \ln|\sigma|\,\mathrm{e}^{-r_2\sigma}\,\mathrm{d}\sigma \\ u_4 &= -\frac{1}{16\pi^2 r_2(r_2^2-r_1^2)(r_2^2-r_3^2)}\int_{-\infty}^{\rho} \sigma^2 \ln|\sigma|\,\mathrm{e}^{r_2\sigma}\,\mathrm{d}\sigma \\ u_5 &= -\frac{1}{16\pi^2 r_3(r_3^2-r_1^2)(r_3^2-r_2^2)}\int_\rho^{\infty} \sigma^2 \ln|\sigma|\,\mathrm{e}^{-r_3\sigma}\,\mathrm{d}\sigma \\ u_6 &= -\frac{1}{16\pi^2 r_3(r_3^2-r_1^2)(r_3^2-r_2^2)}\int_{-\infty}^{\rho} \sigma^2 \ln|\sigma|\,\mathrm{e}^{r_3\sigma}\,\mathrm{d}\sigma. \end{aligned} \tag{32}$$

To simplify eqn (30), we integrate by parts three times for each integral of (32) and substitute into eqn (30). We obtain the expression

$$\Phi(\rho) = -\tfrac{1}{2}\left\{\rho^2 \ln|\rho| \sum_{j=1}^{3} r_j^2\Lambda_j + (2\ln|\rho|+3)\sum_{j=1}^{3}\Lambda_j + \sum_{j=1}^{3}\Lambda_j\left[e^{r_j\rho}\int_{\rho}^{\infty}\frac{e^{-r_j\sigma}}{\sigma}d\sigma - e^{-r_j\rho}\int_{-\infty}^{\rho}\frac{e^{r_j\sigma}}{\sigma}d\sigma\right]\right\} \quad (33)$$

where

$$\Lambda_1 = \frac{1}{4\pi^2 r_1^4(r_1^2-r_2^2)(r_1^2-r_3^2)},\quad \Lambda_2 = \frac{1}{4\pi^2 r_2^4(r_2^2-r_1^2)(r_2^2-r_3^2)},$$
$$\Lambda_3 = \frac{1}{4\pi^2 r_3^4(r_3^2-r_1^2)(r_3^2-r_2^2)}. \quad (34)$$

Notice that r_1 is a real number, but r_2 and r_3 are a couple of conjugate complex variables. Let

$$\eta_1 = (\sqrt{a_3/4}+a_2/4)^{1/2},\quad \eta_2 = (\sqrt{a_3/4}-a_2/4)^{1/2}. \quad (35)$$

From these, eqn (31) can be represented as

$$r_2 = \eta_1 + i\eta_2,\quad r_3 = \eta_1 - i\eta_2. \quad (36)$$

Now define

$$\operatorname{sgn}(r_j) = \begin{cases} 1 & j = 1,2 \\ -1 & j = 3. \end{cases} \quad (37)$$

Then the integrals in eqn (33) can be expressed as (Abramowitz and Stegun, 1966)

$$\int_{\rho}^{\infty}\frac{e^{-r_j\sigma}}{\sigma}d\sigma = E_1(r_j\rho) - \frac{\pi i}{2}\Big(1-\operatorname{sgn}(\rho)\Big)\operatorname{sgn}(r_j)$$
$$\int_{-\infty}^{\rho}\frac{e^{r_j\sigma}}{\sigma}d\sigma = -E_1(-r_j\rho) + \frac{\pi i}{2}\Big(1+\operatorname{sgn}(\rho)\Big)\operatorname{sgn}(r_j)\quad (j = 1,2,3) \quad (38)$$

where $E_1(z)$ is called the Exponential Integral, and its series representation is

$$E_1(z) = -\left[\gamma + \ln z + \sum_{n=1}^{\infty}\frac{(-1)^n z^n}{n n!}\right]\quad (|\arg z| < \pi) \quad (39)$$

where $\gamma = 0.51721$ is Euler's constant. Noting the range of the argument of the complex variable in the Exponential Integral, we can obtain

$$\arg(\pm r_j\rho) = -\Big(\alpha_j \mp \frac{\pi}{2}\operatorname{sgn}(\rho)\Big)\operatorname{sgn}(r_j)\quad (j = 1,2,3) \quad (40)$$

where

$$\alpha_1 = \frac{\pi}{2},\quad \alpha_2 = \alpha_3 = \arctan\left|\frac{\eta_1}{\eta_2}\right|. \quad (41)$$

Therefore, from (38)–(41), we have (Lu, 1989)

$$e^{r_j\rho}\int_\rho^\infty \frac{e^{-r_j\sigma}}{\sigma}\,d\sigma - e^{-r_j\rho}\int_{-\infty}^{\rho}\frac{e^{r_j\sigma}}{\sigma}\,d\sigma = -2\left[\gamma+\ln|r_j\rho|+i\left(\frac{\pi}{2}-\alpha_j\right)\operatorname{sgn}(r_j)\right]$$
$$\times\sum_{m=0}^{\infty}\frac{(r_j\rho)^{2m}}{(2m)!}+2\sum_{m=0}^{\infty}\frac{(r_j\rho)^{2m}}{(2m)!}\psi(2m+1)\quad (j=1,2,3) \tag{42}$$

where

$$\psi(m+1)=\sum_{s=1}^{m}\frac{1}{s},\quad \psi(1)=0. \tag{43}$$

Let $\beta_j = ((\pi/2)-\alpha_j)\operatorname{sgn}(r_j)$; it can be found from (41) that

$$\beta_1=0,\quad \beta_2=\arctan\left|\frac{\eta_2}{\eta_1}\right|,\quad \beta_3=-\beta_2. \tag{44}$$

Substituting (42)–(44) into (33) and deleting the polynomial terms in ρ of degree not greater than 2, we obtain

$$\phi(\rho)=\sum_{j=1}^{3}\Lambda_j\sum_{m=1}^{\infty}\frac{(r_j\rho)^{2m+2}}{(2m+2)!}[\gamma+\ln|r_j\rho|+i\beta_j-\psi(2m+3)] \tag{45}$$

and from (25), we have

$$\Phi(x,y)=\int_0^{2\pi}\phi(\rho)\,d\theta. \tag{46}$$

Thus, the fundamental solution of eqn (20) can be obtained from (45) and (46), in which definite integral (46) should generally be solved by numerical integration.

If a set of generalized unit concentrated forces is applied to a point of the shallow shell in different directions, the corresponding fundamental solutions are those according to eqn (14)

$$\Phi_j(x,y)=\Phi(x,y)/D=\frac{1}{D}\int_0^{2\pi}\phi(\rho)\,d\theta=\int_0^{2\pi}\phi_j(\rho)\,d\theta\quad (j=1,\ldots,5) \tag{47}$$

where $\phi_j(\rho)=\phi(\rho)/D$. Therefore, the displacements and stresses in the shell caused by the set of the unit concentrated forces can be expressed according to (9) and (15)

$$\{\mathbb{U}\}=[\mathbb{F}]\{\Phi^*\}=[\mathbb{F}]\int_0^{2\pi}\{\phi^*\}\,d\theta=\int_0^{2\pi}[\tilde{\mathbb{F}}]\{\phi^*\}\,d\theta=\int_0^{2\pi}\{\tilde{\mathbb{U}}\}\,d\theta \tag{48}$$

$$\{\mathbb{T}\}=[\mathbb{R}]\{\Phi^*\}=\int_0^{2\pi}[\tilde{\mathbb{R}}]\{\phi^*\}\,d\theta=\int_0^{2\pi}[\tilde{\mathbb{T}}]\,d\theta \tag{49}$$

where

$$\{\phi^*\}=[\phi_1,\phi_2,\ldots,\phi_5]^T \tag{50}$$

and $[\tilde{\mathbb{F}}]$, $[\tilde{\mathbb{R}}]$ are differential operator matrices composed by differential operators $d^k/d\rho^k$ ($k=1,2,\ldots,9$). The elements of the matrices can be obtained simply by replacing the

d^k/dx^k, d^k/dy^k of corresponding elements in matrices [$\mathbb{F}$], [$\mathbb{R}$] by $\omega_1^k(d^k/d\rho^k)$, $\omega_2^k(d^k/d\rho^k)$, and are given in the Appendix.

COMPUTATIONAL METHOD FOR THE FUNDAMENTAL SOLUTIONS

Representations for derivatives of $\phi(\rho)$

As discussed before, there are derivatives of $\phi(\rho)$ for variable ρ up to ninth order in matrices [$\tilde{\mathbb{F}}$] and [$\tilde{\mathbb{R}}$]. Substituting (38) into (33), $\phi(\rho)$ can be expressed in another form

$$\phi(\rho) = -\tfrac{1}{2}\left\{\rho^2 \ln|\rho| \sum_{j=1}^{3} r_j^2 \Lambda_j + (2\ln|\rho|+3)\sum_{j=1}^{3}\Lambda_j + \sum_{j=1}^{3}\Lambda_j \chi_j\right\} \tag{51}$$

where

$$\chi_j(r_j\rho) = e^{r_j\rho} E_1(r_j\rho) + e^{-r_j\rho} E_1(-r_j\rho) - i\pi[\mathrm{ch}\,(r_j\rho) - \mathrm{sh}\,(r_j\rho)\,\mathrm{sgn}\,(\rho)]\,\mathrm{sgn}\,(r_j) \quad (j = 1, 2, 3). \tag{52}$$

According to Abramowitz and Stegum (1966), we have

$$\frac{d^n}{dz^n}[e^z E_1(z)] = \frac{d^{n-1}}{dz^{n-1}}[e^z E_1(z)] + \frac{(-1)^n(n-1)!}{z^n} \quad (n = 1, 2, \ldots). \tag{53}$$

Thus, we obtain

$$\frac{d\chi_j(r_j\rho)}{d\rho} = r_j\lambda_j(r_j\rho) - \frac{2}{\rho} \quad (j = 1, 2, 3) \tag{54}$$

where

$$\lambda_j(r_j\rho) = e^{r_j\rho} E_1(r_j\rho) - e^{-r_j\rho} E_1(-r_j\rho) - i\pi[\mathrm{sh}\,(r_j\rho) - \mathrm{ch}\,(r_j\rho)\,\mathrm{sgn}\,(\rho)]\,\mathrm{sgn}\,(r_j) \quad (j = 1, 2, 3) \tag{55}$$

and the following relations are also true

$$\frac{d\lambda_j(r_j\rho)}{d\rho} = r_j\chi_j(r_j\rho) \quad (j = 1, 2, 3). \tag{56}$$

From eqn (34), we can obtain the relations

$$\sum_{j=1}^{3} r_j^2\Lambda_j = \frac{1}{4\pi^2}\frac{1}{r_1^2 r_2^2 r_3^2}, \quad \sum_{j=1}^{3} r_j^4\Lambda_j = \sum_{j=1}^{3} r_j^6\Lambda_j = 0, \quad \sum_{j=1}^{3} r_j^8\Lambda_j = \frac{1}{4\pi^2}. \tag{57}$$

Now, from (51)–(57), we can obtain the derivatives of $\phi(\rho)$

$$\frac{d\phi}{d\rho} = -\tfrac{1}{2}\left\{\rho(2\ln|\rho|+1)\sum_{j=1}^{3} r_j^2\Lambda_j + \sum_{j=1}^{3}\Lambda_j r_j \lambda_j\right\},$$

$$\frac{d^2\phi}{d\rho^2} = -\tfrac{1}{2}\left\{\rho(2\ln|\rho|+3)\sum_{j=1}^{3} r_j^2\Lambda_j + \sum_{j=1}^{3}\Lambda_j r_j^2 \chi_j\right\},$$

$$\frac{d^{2k+1}}{d\rho^{2k+1}} = -\tfrac{1}{2}\sum_{j=1}^{3}\Lambda_j r^{2k+1}\lambda_j, \quad \frac{d^{2(k+1)}\phi}{d\rho^{2(k+1)}} = -\tfrac{1}{2}\sum_{j=1}^{3}\Lambda_j r^{2(k+1)}\chi_j,$$

$$\frac{d^9\phi}{d\rho^9} = -\tfrac{1}{2}\sum_{j=1}^{3}\Lambda_j r^9\lambda_j + \frac{1}{4\pi^2}\frac{1}{\rho} \quad (k = 1, 2, 3). \tag{58}$$

Therefore, the problems of calculating the derivative values of the fundamental solutions are reduced to evaluating the values of χ_j and λ_j $(j = 1, 2, 3)$.

Computation of χ_j and λ_j

Because r_2 and r_3, Λ_2 and Λ_3 are complex conjugate quantities, χ_2 and χ_3, λ_2 and λ_3 are also complex conjugate functions from (52) and (55). Therefore, only χ_1, λ_1, χ_2 and λ_2 should be calculated in the numerical computations.

As r_1 is a positive real number, according to the properties of the exponential integral and (52) and (55), we have

$$\chi_1 = \begin{cases} e^{r_1\rho} E_1(r_1\rho) - e^{-r_1\rho}\,\mathrm{Ei}(r_1\rho) & \rho > 0 \\ -e^{r_1\rho}\,\mathrm{Ei}(|r_1\rho|) + e^{-r_1\rho} E_1(|r_1\rho|) & \rho < 0 \end{cases} \tag{59}$$

$$\lambda_1 = \begin{cases} e^{r_1\rho} E_1(r_1\rho) + e^{-r_1\rho}\,\mathrm{Ei}(r_1\rho) & \rho > 0 \\ -e^{r_1\rho}\,\mathrm{Ei}(|r_1\rho|) - e^{-r_1\rho} E_1(|r_1\rho|) & \rho < 0 \end{cases} \tag{60}$$

where exponential integrals $E_1(x)$ and $\mathrm{Ei}(x)$ can be evaluated using the approximate formulas of Cody and Thacher (1968, 1969).

Because r_2 is a complex variable, the numerical computations of the complex functions χ_2 and λ_2 are comparatively difficult. By using the series expansions of exponential integrals and hyperbolic functions, eqns (52) and (55) can be expressed as

$$\chi_j = -2\sum_{m=0}^{\infty} \frac{(r_j\rho)^{2m}}{(2m)!}[\gamma + \ln|r_j\rho| + i\beta_j - \psi(2m+1)] \tag{61}$$

$$\lambda_j = -2\sum_{m=0}^{\infty} \frac{(r_j\rho)^{2m-1}}{(2m-1)!}[\gamma + \ln|r_j\rho| + i\beta_j - \psi(2m)]. \tag{62}$$

If $|r_2\rho|$ is small, the convergence rate is fast by using the above equations for the evaluations of χ_2 and λ_2, but the computing time will increase when $|r_2\rho|$ is large. In this case, we can use the following expressions:

$$\chi_2 = \begin{cases} e^{r_2\rho} E_1(r_2\rho) + e^{-r_2\rho} E_1(-r_2\rho) - i\pi\, e^{-r_2\rho} & \rho > 0 \\ e^{r_2\rho} E_1(r_2\rho) + e^{-r_2\rho} E_1(-r_2\rho) + i\pi\, e^{r_2\rho} & \rho < 0 \end{cases} \tag{63}$$

$$\lambda_2 = \begin{cases} e^{r_2\rho} E_1(r_2\rho) - e^{-r_2\rho} E_1(-r_2\rho) + i\pi\, e^{-r_2\rho} & \rho > 0 \\ e^{r_2\rho} E_1(r_2\rho) - e^{-r_2\rho} E_1(-r_2\rho) - i\pi\, e^{r_2\rho} & \rho < 0 \end{cases} \tag{64}$$

where $e^{r_2\rho} E_1(r_2\rho)$ and $e^{-r_2\rho} E_1(-r_2\rho)$ can be computed by using the following approximate formulas (Abramowitz and Stegum, 1966):

when $x > 10$ or $y > 10$ $(z = x+iy)$

$$e^{z} E_1(z) = \frac{0.711093}{z+0.415775} + \frac{0.278518}{z+2.29428} + \frac{0.010389}{z+6.2900} \tag{65}$$

when $|z| > 15$ $(x \ngtr 10, y \ngtr 10)$

$$e^{z} E_1(z) = \frac{1}{z}\left\{1 - \frac{1}{z} + \frac{1\cdot 2}{z^2} - \frac{1\cdot 2\cdot 3}{z^3} + \cdots\right\} \tag{66}$$

when $4 \leqslant |z| \leqslant 15$ $(x \ngtr 10, y \ngtr 10)$

$$e^{z} E_1(z) = \left(\frac{1}{z+}\,\frac{1}{1+}\,\frac{1}{z+}\,\frac{2}{1+}\,\frac{2}{z+}\,\frac{3}{1+}\,\frac{3}{z+}\cdots\right). \tag{67}$$

When $|z| < 4$, (61) and (62) can be used to evaluate χ_2 and λ_2. Once χ_1, χ_2, λ_1 and λ_2 have been obtained, the derivatives of $\phi(\rho)$ can be easily calculated by eqn (58). For example,

$$\frac{d^{2k+1}\phi}{d\rho^{2k+1}} = -\tfrac{1}{2}[\Lambda_1 r_1^{2k+1}\lambda_1 + 2\mathrm{Re}\,(\Lambda_2 r_2^{2k+1}\lambda_2)]$$

$$\frac{d^{2(k+1)}\phi}{d\rho^{2(k+1)}} = -\tfrac{1}{2}[\Lambda_1 r_1^{2(k+1)}\chi_1 + 2\mathrm{Re}\,(\Lambda_2 r_2^{2(k+1)}\chi_2)] \quad (k = 1, 2, 3);$$

the other derivatives of $\phi(\rho)$ can be obtained in a similar way.

Computations of the fundamental solutions

To calculate the fundamental solutions, the following definite integrals are encountered:

$$\frac{\partial^{k+l}\Phi(x,y)}{\partial x^k\,\partial y^l} = \int_0^{2\pi} \cos^k\theta \sin^l\theta \frac{d^{k+l}\phi(\rho)}{d\rho^{k+l}}\,d\theta \quad (k, l = 0, 1, 2, \ldots). \tag{68}$$

This can be treated by the Gaussian numerical integration method. The interval $(0, 2\pi)$ is divided into sub-regions according to the given precision, and each sub-region is interpolated by five Gaussian integral points. Numerical results show that this treatment is suitable.

When Gaussian curvatures of the shells are negative, it can be seen from (28) and (31) that r_2 will be zero for some values of θ. In these cases, a singularity exists in the integrals. To avoid this situation, the values of θ in which r_2 becomes zero should be determined in advance.

To do this, supposing that the values to make r_2 or a_2 zero are θ_0, we have from (28)

$$k_1 \sin^2\theta_0 + k_2\cos^2\theta_0 = 0$$

so

$$\tan^2\theta_0 = -\frac{k_2}{k_1} = \left|\frac{k_2}{k_1}\right| \quad \text{or} \quad \tan\theta_0 = \pm\sqrt{\left|\frac{k_2}{k_1}\right|}.$$

This gives

$$\theta_0 = \begin{cases} \arctan\sqrt{|k_2/k_1|} \\ \pi \pm \arctan\sqrt{|k_2/k_1|} \\ 2\pi - \arctan\sqrt{|k_2/k_1} \end{cases}. \tag{69}$$

Therefore, in the case of $k_1k_2 \leqslant 0$, the region $(0, 2\pi)$ can be divided into four sub-regions, and numerical calculations are carried out in these sub-regions. The problems of singularity can be avoided in this way. The calculation practice shows that this treatment is suitable.

NUMERICAL RESULTS

To check the fundamental solutions obtained and treatment programme discussed before, we selected the case of a concentrated normal force acting on infinite shallow shells with variable $\tau = k_2/k_1$ and evaluated the distributions of generalized internal forces and displacements of the shell along $x = 0$ and $y = 0$ respectively. These results are presented graphically in Figs 1–14, in which the horizontal coordinate is the dimensionless value

$$\beta r\left(\beta r = \left[\frac{\sqrt{12(1-\nu^2)}k_1}{h}\right]^{1/2}\sqrt{x^2+y^2}\right),$$

and the vertical one represents dimensionless generalized forces and displacements respectively.

As $-1 \leqslant \tau \leqslant 1$, the value $y = 0$ represents the line of maximum curvature, while $x = 0$ represents the minimum one. Since the displacements and the forces have singularity of order ln r at the point where the concentrated force is applied, the numerical calculations begin from $\beta r = 0.01$.

From Figs 1–12, we can see that the trends of the internal forces and the displacements due to βr and τ are similar to those obtained for thin shell structures (Matsui and Matsuoka, 1978). This can be explained by the fact that the theory of the shallow shell involving shear deformation would tend to thin shell theory when the thickness of the shell is reduced gradually. Figures 13 and 14 show the variation of ψ_x along $y = 0$ and ψ_y along $x = 0$ with the variables βr and τ, which cannot be obtained from thin shell theory. It can be seen from the figures that ψ_x and ψ_y are influenced strongly by the shell geometry, especially for cylindrical shells.

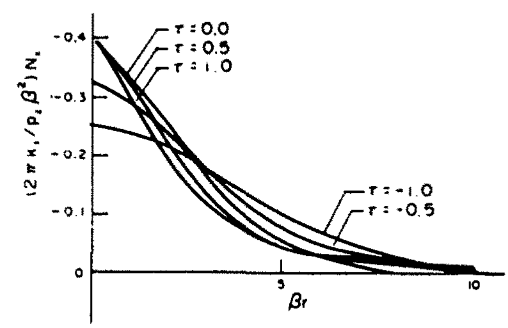

Fig. 1. Membrane stress resultant N_x ($y = 0$, $\tau = k_2/k_1$).

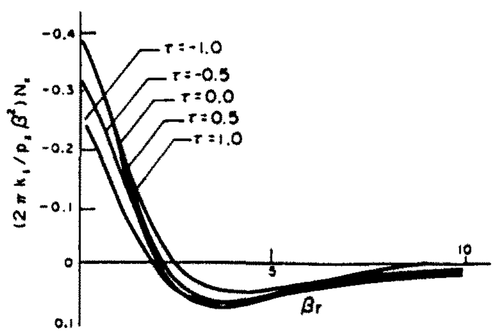

Fig. 2. Membrane stress resultant N_x ($x = 0$, $\tau = k_2/k_1$).

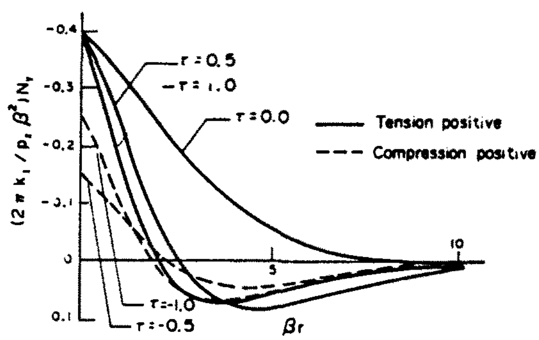

Fig. 3. Membrane stress resultant N_y ($y = 0$, $\tau = k_2/k_1$).

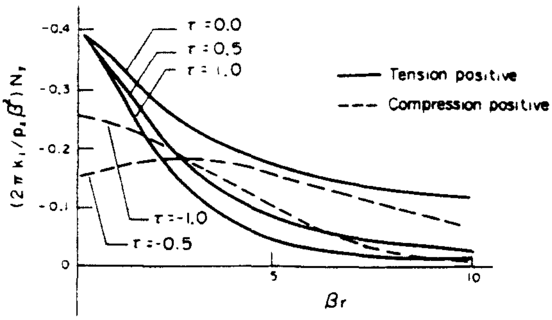

Fig. 4. Membrane stress resultant N_r ($x = 0$, $\tau = k_2/k_1$).

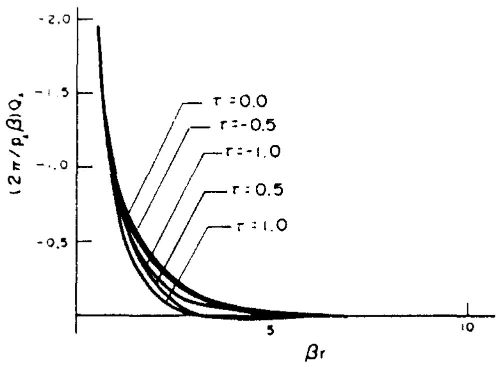

Fig. 5. Transverse shear force Q_x ($y = 0$, $\tau = k_2/k_1$).

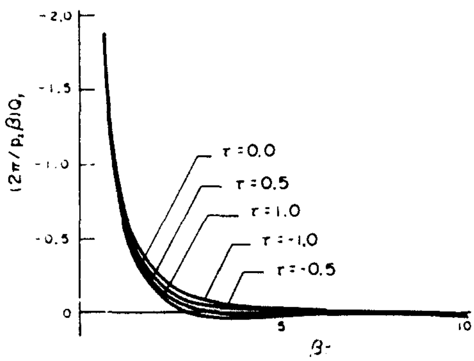

Fig. 6. Transverse shear force Q_y ($x = 0$, $\tau = k_2/k_1$).

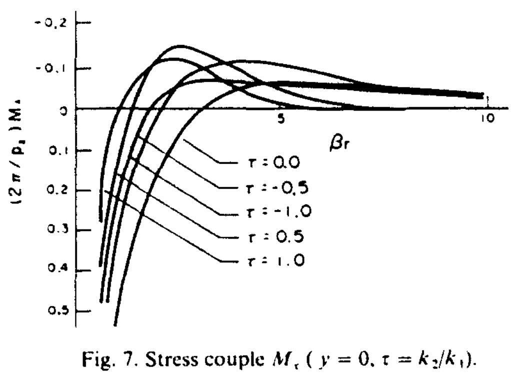

Fig. 7. Stress couple M_x ($y = 0$, $\tau = k_2/k_1$).

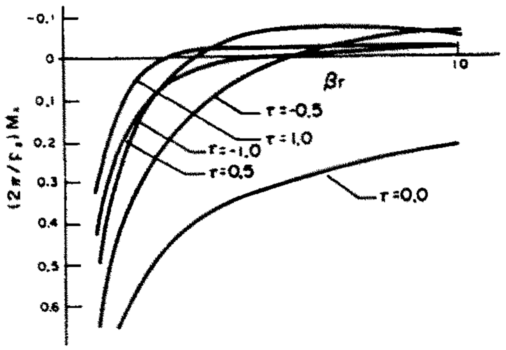

Fig. 8. Stress couple M_x ($x = 0$, $\tau = k_2/k_1$).

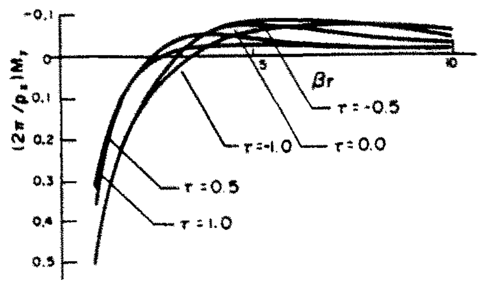

Fig. 9. Stress couple M_y ($y = 0$, $\tau = k_2/k_1$).

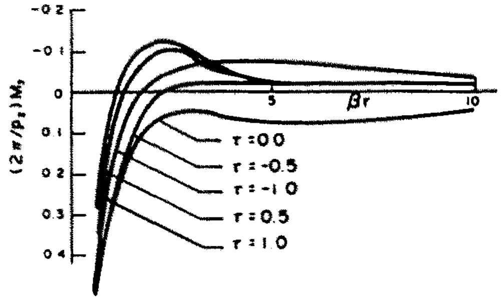

Fig. 10. Stress couple M_y ($x = 0$, $\tau = k_2/k_1$).

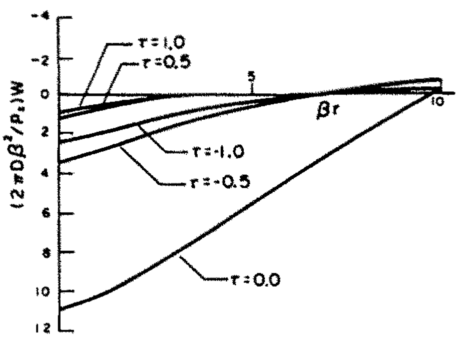

Fig. 11. Normal deflection W ($y = 0$, $\tau = k_2/k_1$).

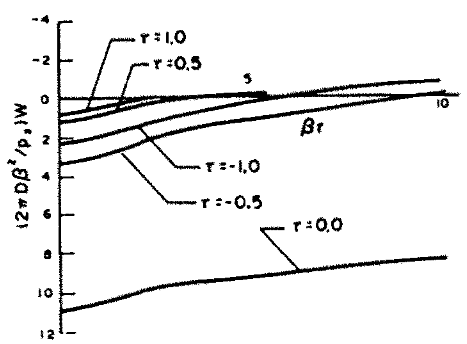

Fig. 12. Normal deflection W ($x = 0$, $\tau = k_2/k_1$).

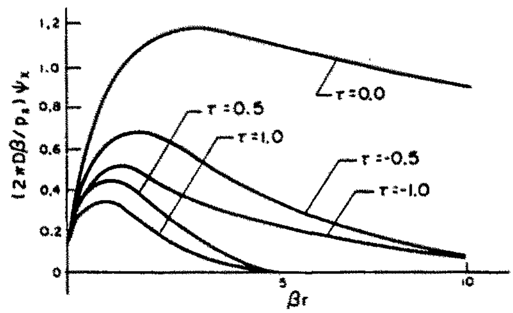

Fig. 13. Rotation ψ_x ($y = 0$, $\tau = k_2/k_1$).

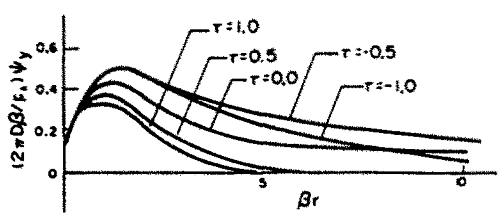

Fig. 14. Rotation ψ_y ($x = 0$, $\tau = k_2/k_1$).

CONCLUSIONS

In this paper, the governing equations of the shallow shell involving shear deformation are decoupled and a set of equations for displacement functions are derived. Representations of this form are very useful in the study of this kind of shells. The fundamental solutions of the shells are then obtained and the corresponding computational methods are discussed in detail.

The boundary element analyses for the shallow shell involving shear deformation by using the fundamental solution obtained have been completed (Lu, 1989) and will be given in a subsequent paper.

Acknowledgement—The support by the National Natural Science Foundation of China is gratefully acknowledged.

REFERENCES

Abramowitz, M. and Stegum, I. A. (Eds) (1966). *Handbook of Mathematical Functions*. Fifth Printing, New York.

Cody, W. J. and Thacher, H. C. (1968). Rational Chebyshev approximations for the exponential integral $E_1(x)$. *Math. Comp.* **22**, 641–649.

Cody, W. J. and Thacher, H. C. (1969). Chebyshev approximation for the exponential integral $Ei(x)$. *Math. Comp.* **23**, 289–303.

Flügge, W. and Elling, R. E. (1972). Singular solutions for shallow shells. *Int. J. Solids Structures* **8**, 227–247.

Gel'fand, I. M. and Shilov, G. E. (1966). *Generalized Functions*, Vol. 1. Academic Press, New York.

Hörmander, L. (1963). *Linear Partial Differential Operators*. Springer, Berlin.

Jahanshahi, A. (1964). Some notes on singular solutions and the Green's functions in the theory of plates and shells. *J. Appl. Mech.* **31**, 441–446.

Lei, X. Y. and Huang, M. G. (1987). Boundary element method for shallow spherical shell bending problems involving shear deformation. *Boundary Element Methods in Engineering, 9th Int. Conf. on BEM*.

Lu, P. (1989). Boundary element and complex variable analytic methods for solving plate and shell problems. Ph.D. dissertation, University of Science and Technology of China [in Chinese].

Matsui, T. and Matsuoka, O. (1978). The fundamental solution in the theory of shallow shells. *Int. J. Solids Structures* **14**, 971–986.

Sanders, J. L., Jr. (1970). Singular solution to the shallow shell equations. *J. Appl. Mech.* **37**, 361–366.

Sih, G. C. (Ed.) (1977). *Mechanics of Fracture*, Vol. 3. Noordhoff, Amsterdam.

Simmonds, J. G. and Bradley, M. R. (1976). The fundamental solution for a shallow shell with an arbitrary quadratic midsurface. *J. Appl. Mech.* **43**, 286–290.

Ye, N. X. and Xu, H. Y. (1978). *Ordinary Differential Equations*. China Academic Publishers, Beijing [in Chinese].

APPENDIX

The elements of differential operator matrix $[\mathbb{R}]$ are

$$R_{1j} = B[F_{1j}D_1 + vF_{2j}D_2 + (k_1 + vk_2)F_{3j}],\quad R_{2j} = B[vF_{1j}D_1 + F_{2j}D_2 + (k_2 + vk_1)F_{3j}],$$

$$R_{3j} = \tfrac{1}{2}(1-v)B[F_{1j}D_2 + F_{2j}D_1],\quad R_{4j} = C[F_{3j}D_1 + F_{4j}],\quad R_{6j} = C[F_{3j}D_2 + F_{5j}],$$

$$R_{6j} = D[F_{4j}D_1 + vF_{5j}D_2],\quad R_{7j} = D[vF_{4j}D_1 + F_{5j}D_2],\quad R_{8j} = \tfrac{1}{2}(1-v)D[F_{4j}D_2 + F_{5j}D_1]\quad (j = 1,2,\ldots,5)$$

where D_1 and D_2 are given by (7), and F_{ij} by (18).

The elements of operator matrices $[\tilde{\mathbb{F}}]$ and $[\tilde{\mathbb{R}}]$ are defined as

$$D_\rho = \frac{\mathrm{d}}{\mathrm{d}\rho},\quad D_\rho^k = \frac{\mathrm{d}^k}{\mathrm{d}\rho^k},\quad \tau = \frac{k_2}{k_1},$$

$$k_1 = \omega_1^2 + \frac{2}{1-v}\omega_2^2,\qquad k_2 = \omega_2^2 + \frac{2}{1-v}\omega_1^2,$$

$$k_3 = k_1^2[(1+\tau+2v\tau)\omega_1^2 + 2(1+v)\omega_2^2],\quad k_4 = k_1^2[(1+\tau+2v\tau)\omega_2^2 + 2(1+v)\omega_2^2\tau^2],$$

$$k_5 = k_1[(1+v\tau)\omega_1^3 + (2+v-\tau)\omega_1\omega_2^2],\quad k_6 = k_1[(\tau+v)\omega_2^3 + ((2+v)\tau-1)\omega_1^2\omega_2].$$

Therefore,

$$\tilde{F}_{11} = \frac{k_1}{B}D_\rho^8 - \left(\frac{k_3}{C} + \frac{k_1}{B}a_1^2\right)D_\rho^6 + k_3\left(\frac{1}{D} + \frac{a_1^2}{C}\right)D_\rho^4 - \frac{k_3}{D}a_1^2 D_\rho^2,$$

$$\tilde{F}_{12} = \tilde{F}_{21} = \omega_1\omega_2\left[-\frac{1}{B}\frac{1+v}{1-v}D_\rho^8 + \left(\frac{1}{B}\frac{1+v}{1-v} + (1-\tau)^2\frac{k_1^2}{C}\right)D_\rho^6 - k_1^2(1-\tau)^2\left(\frac{1}{D} + \frac{a_1^2}{C}\right)D_\rho^4 + \frac{1}{D}k_1^2(1-\tau)^2 a_1^2 D_\rho^2\right],$$

$$\tilde{F}_{13} = \tilde{F}_{31} = \frac{k_5}{C}\left[D_\rho^7 - \left(a_1^2 + \frac{C}{D}\right)D_\rho^5 + \frac{C}{D}a_1^2 D_\rho^3\right],\quad \tilde{F}_{14} = \tilde{F}_{41} = \frac{k_5}{D}\omega_1[D_\rho^6 - a_1^2 D_\rho^4],$$

$$\tilde{F}_{15} = \tilde{F}_{51} = \frac{k_5}{D}\omega_2[D_\rho^6 - a_1^2 D_\rho^4],\quad \tilde{F}_{22} = \frac{k_2}{B}D_\rho^8 - \left(\frac{k_4}{C} + \frac{k_2}{B}a_1^2\right)D_\rho^6 + k_4\left(\frac{1}{D} + \frac{a_1^2}{C}\right)D_\rho^4 - \frac{k_4}{D}a_1^2 D_\rho^2,$$

$$\tilde{F}_{23} = \tilde{F}_{32} = \frac{k_6}{C}\left[D_\rho^7 - \left(a_1^2 + \frac{C}{D}\right)D_\rho^5 + \frac{C}{D}a_1^2 D_\rho^3\right].$$

$$\tilde{F}_{24} = \tilde{F}_{42} = \frac{k_6}{D}\omega_1[D_\rho^6 - a_1^2 D_\rho^4], \quad \tilde{F}_{25} = \tilde{F}_{52} = \frac{k_6}{D}\omega_2[D_\rho^6 - a_1^2 D_\rho^4],$$

$$\tilde{F}_{33} = -\frac{1}{C}\left[D_\rho^8 - \left(a_1^2 + \frac{C}{D}\right)D_\rho^6 + \frac{C}{D}a_1^2 D_\rho^4\right],$$

$$\tilde{F}_{34} = \tilde{F}_{43} = -\frac{1}{D}\omega_1[D_\rho^7 - a_1^2 D_\rho^5], \quad \tilde{F}_{35} = \tilde{F}_{53} = -\frac{1}{D}\omega_2[D_\rho^7 - a_1^2 D_\rho^5],$$

$$\tilde{F}_{44} = \frac{1}{D}[k_1 D_\rho^8 - (a_1^2\omega_1^2 + a_2 k_1)D_\rho^6 + a_1^2 a_2 D_\rho^4], \quad \tilde{F}_{45} = \tilde{F}_{54} = \frac{1}{D}\omega_1\omega_2\left[-\frac{1+\nu}{1-\nu}D_\rho^8 + \left(\frac{1+\nu}{1-\nu}a_2 - a_1^2\right)D_\rho^6\right],$$

$$\tilde{F}_{55} = \frac{1}{D}[k_2 D_\rho^8 - (a_1^2\omega_2^2 + a_2 k_2)D_\rho^6 + a_1^2 a_2 D_\rho^4],$$

where a_1, a_2 are given by (28), and ω_1 and ω_2 are given by (22).

The components of operator matrix $[\mathbb{R}]$ are

$$\tilde{R}_{1j} = B[\omega_1 \tilde{F}_{1j} D_\rho + \nu\omega_2 \tilde{F}_{2j} D_\rho + k_1(1+\nu\tau)\tilde{F}_{3j}],$$
$$\tilde{R}_{2j} = B[\nu\omega_1 \tilde{F}_{1j} D_\rho + \omega_2 \tilde{F}_{2j} D_\rho + k_1(\tau+\nu)\tilde{F}_{3j}],$$
$$\tilde{R}_{3j} = \tfrac{1}{2}(1-\nu)B[\omega_2 \tilde{F}_{1j} + \omega_1 \tilde{F}_{2j}]D_\rho, \quad \tilde{R}_{4j} = C[\omega_1 \tilde{F}_{3j} D_\rho + \tilde{F}_{4j}],$$
$$\tilde{R}_{5j} = C[\omega_2 \tilde{F}_{3j} D_\rho + \tilde{F}_{5j}], \quad \tilde{R}_{6j} = D[\omega_1 \tilde{F}_{4j} + \nu\omega_2 \tilde{F}_{5j}]D_\rho,$$
$$\tilde{R}_{7j} = D[\nu\omega_1 \tilde{F}_{4j} + \omega_2 \tilde{F}_{5j}]D_\rho, \quad \tilde{R}_{8j} = \tfrac{1}{2}(1-\nu)D[\omega_2 \tilde{F}_{4j} + \omega_1 \tilde{F}_{5j}]D_\rho. \quad (j = 1, 2, \ldots, 5)$$